Numerical Analysis Qualifying Exam
Mathematics Department, New Mexico Tech
January 10, 2014

(Answer 5 of the 6 questions. You must explicitly state which answers you want graded. If you do not explicitly state which answers you want graded, then we will grade all six answers and your grade will be sum of the five lowest grades)

1. Let $f(x) = \frac{1-\sin x}{\frac{\pi}{2}-x}$

   (a) Explain why straightforward evaluation of $f(x)$ in floating point double precision arithmatic produces wildly inaccurate answers for $x \approx \frac{\pi}{2}$.

   (b) Derive an alternative formula that is much more accurate for $x \approx \frac{\pi}{2}$.

   (c) Demonstrate your answers in parts a) and b) by computing relative errors of the numerical formulas using the 3 digit chopping floating point arithmetics and $x = 1.56$

2. Find coefficients A and B in the open Newton-Cotes formula

$$\int_0^1 f(x)dx \approx Af\left(\frac{1}{3}\right) + Bf\left(\frac{2}{3}\right)$$

   Transform this into a formula for $\int_a^b f(x)dx$.

3. Let V be an inner product space with inner product $\langle\cdot,\cdot\rangle$. Let $\{v_i\}_{i=1}^n$ be a vector sequence in V.

   (a) Describe and formulate the Gram-Schmidt process applied to the sequence $\{v_i\}$.

   (b) Let V be the space P_2 of polynomials of degree at most two. Let $\langle f,g\rangle_2 = \int_0^1 f(t)\,g(t)\,dt$ be the inner product on V. Apply the Gram-Schmidt process to the sequence $\{1,t,t^2\}$.

   (c) Find a quadratic polynomial $p(t)$ that approximates the function $f(t) = t^3$ on the interval $[0,1]$ in the best least squares sense relative to the inner procutct $(\cdot,\cdot)_2$, that is find $q(t)$ satisfying

$$\int_0^1 (f(t) - q(t))^2\,dt = \min_{r(t)\in P_2} \int_0^1 (f(t) - r(t))^2\,dt$$

4. Let $n > 1$ be an integer, let $[a, b]$ be a line interval, and let $h = \frac{b-a}{n}$. Let values of a function $f(x)$ be known at the equidistant points $x_i = a + ih$ for $i = 0, 1, \ldots, n$. In addition, assume that the value $f'(a)$ is also known.

   (a) Define a smooth piecewise quadratic function $s(x)$ that interpolates the function $f(x)$ at the points $\{x_i\}_{i=0}^{n}$. That is, function $s(x)$ must:

      i. be a quadratic polynomial on every subinterval $[x_i, x_{i+1}]$
      ii. be continuously differentiable on the interval $[a, b]$
      iii. interpolate $f(x)$ at points $\{x_i\}_{i=0}^{n}$
      iv. satisy an additional condition $s'(a) = f'(a)$.

      To this end, specify the set of variables and equations that identity the function $s(x)$.

   (b) Give a pseudo-code of an implementation of the algorithm.

   (c) Find the piecewise quadratic interpolant of function $f(x) = \sin(x)$ on the interval $[0, \pi]$ with $n = 2$. Round your calculations to three decimal places.

5. Let $A \in R^{n \times n}$ be nonsingular. Assume that $A = LU$ and $A = \tilde{L}\tilde{U}$ are two LU decompositions of A where L and $\tilde{L}$ are unit lower triangular matrices and U and $\tilde{U}$ are upper triangular matrices.

   (a) Show that $\tilde{L}^{-1}L = \tilde{U}U^{-1}$. What does this say about the matrices $\tilde{L}^{-1}L$ and $\tilde{U}U^{-1}$?

   (b) Write $L = \mathcal{L} + I$ and $\tilde{L} = \tilde{\mathcal{L}} + I$ where $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are strictly lower triangular and using part a), show that $\mathcal{L} = \tilde{\mathcal{L}}$. This shows that $L = \tilde{L}$.

   (c) Show that $U = \tilde{U}$. Thus the LU decomposition of A is unique.

6. Let $A \in R^{n \times n}$ be semi-simple, i.e. it has a complete set of eigenvectors, $A\bar{q}_j = \lambda_j \bar{q}_j$, $\bar{q}_k \in R^n$, $\|\bar{q}_j\| = 1$, $\lambda_j \in \mathbb{C}$, $j = 1, 2, \ldots, n$, where $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$, and consider the iteration

$$\bar{x}_{k+1} = \frac{1}{\lambda_1} A \bar{x}_k$$

   where $\bar{x}_0 \in R^n$ is a randomly chosen vector. Show that $\bar{x}_k$ converges to a multiple of $\bar{q}_1$ and that the convergence is linear. What condition must be true for the sequence $\tilde{x}_k = \frac{1}{\|\bar{x}_k\|}\bar{x}_k$ to converge to $\bar{q}_1$?