Ph.D. Preliminary Examination in Numerical Analysis
Department of Mathematics
New Mexico Institute of Mining and Technology
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1. This exam is four hours long.
2. You need a scientific calculator for this exam.
3. Work out all six problems.
4. Start the solution of each problem on a new page.
5. Number all of your pages.
6. Sign your name on the following line and put the total number of pages.
7. Use this sheet as a coversheet for your papers.

NAME: ____ **No. of pages:**___________

Problem 1. Consider the equation

$$x + \ln x = 0.$$

This equation has a solution somewhere near $x = 0.55$.

1. Derive a convergent fixed point iteration, $x_{n+1} = F(x_n)$ for solving this equation. Find an interval such that if x_0 is in this interval, the fixed point iteration will converge to the root.
2. Starting with $x_0 = 0.55$, use your iteration to solve the equation, obtaining a root accurate to 3 digits.

Problem 2. Describe the Taylor series method of order m for solving the initial value problem

$$x' = f(t, x), \quad t > 0, \quad x(0) = x_0.$$

What is the order of the local truncation error of the Taylor series method of order m? Give the explicit formula of the method for the case $m = 2$ applied to the model problem. Apply the Taylor series method of order two to compute the numerical approximation

$$X_2 \approx x(t_2)$$

for the problem with $f(t, x) = x \cos t$, $x_0 = 1$, and the step size $h = 0.1$. Find the exact solution and compute the relative error for X_2.

Problem 3. Let

$$f(x) = \cosh(x), \quad -1 \leq x \leq 1.$$

Suppose that we interpolate this function using 15 points in the interval and a polynomial of degree 14. Find a numerical bound on the maximum absolute error over the interval $-1 \leq x \leq 1$.

Problem 4. State and prove existence and uniqueness statement for the Cholesky decomposition of a positive definite matrix.

Hints.

1. Prove existence by mathematical induction. Consider the matrix partitioning

$$A = \begin{pmatrix} a_{11} & \bar{a}^t \\ \bar{a} & \hat{A} \end{pmatrix}, \qquad a_{11} \in R.$$

Use the factorization

$$A = \begin{pmatrix} r_{11} & \bar{0}^t \\ \bar{r} & I \end{pmatrix} \begin{pmatrix} 1 & \bar{0}^t \\ \bar{0} & \hat{A} - \bar{r}\bar{r}^t \end{pmatrix} \begin{pmatrix} r_{11} & \bar{r}^t \\ \bar{0} & I \end{pmatrix}, \tag{1}$$

where $r_{11} = \sqrt{a_{11}}$, $\bar{r} = (1/r_{11})\bar{a}$, $\bar{0}$ is the zero vector, and I is the identity matrix.

2. Prove uniqueness by mathematical induction. Suppose

$$A = R^t R = G^t G,$$

and consider appropriate partitionings of the matrices A, R, and G.

3. You may use some required properties of positive definite matrices without giving their proofs.

Problem 5. Describe the main steps of Francis's Algorithm of degree one (also known as implicitly shifted QR algorithm) for computing eigenvalues and eigenvectors of a proper Hessenberg matrix $A \in C^{n\times n}$. Describe how the Rayleigh and the Wilkinson shifts are selected. How is the convergence of the method determined with these shifts? A pseudocode of the algorithm is not required.

Problem 6. Consider the steepest descent method for solving a system of equations $Ax = b$, where A is symmetric and positive definite. In k-th iteration, let

$$r^{(k)} = b - Ax^{(k)}$$

be the residual vector. We update the solution with

$$x^{(k+1)} = x^{(k)} + tr^{(k)},$$

where

$$t = \frac{r^{(k)T} r^{(k)}}{r^{(k)T} A r^{(k)}}.$$

Show that $r^{(k+1)}$ is orthogonal to $r^{(k)}$.