Probability and Statistics, Sample Prelim IV Questions, Fall 2021

1. Let $X_1, ..., X_n$ be a random sample from a distribution with probability density function

$$f(x \mid \theta) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\theta^{\alpha}} e^{-x/\theta}, \quad \theta > 0, \text{ with known } \alpha = 2$$

(a) Find the maximum likelihood estimator (MLE) of θ

(b) Show that the method of moments estimator (MoM) of θ is the same as the MLE in part (a).

(c) Let $\tau = 1/\theta$. Find the MLE of τ (denote it as $\hat{\tau}$). Is $\hat{\tau}$ consistent? State the properties/results you are using when answering this question.

2. Let $X_1, ..., X_n$ be independent random variables where $X_i \sim \texttt{Poisson}(\theta)$, $\theta > 0$ is unknown,

$$f(x|\theta) = \begin{cases} e^{-\theta} \frac{\theta^x}{x!} & \text{for } x = 0, 1, 2 \dots \\ 0 & \text{elsewhere} \end{cases}$$

(a) Determine the Cramer-Rao lower bound for the variance of unbiased estimators of θ.

(b) Find the minimum variance unbiased estimator of θ. Also, find the variance of this estimator.

(c) Let the prior density for θ be exponential distribution with mean 1 (i.e., $f(\theta) = e^{-\theta}$). Find an explicit expression for a Bayes estimate of θ.

3. Consider the following joint density for random variables X and Y:

$$f(x, y) = \begin{cases} kx, & \text{for } 0 < x, y < 1 \text{ and } x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find k.

(b) Find the marginal density of X. Are these random variables independent?

(c) Find the conditional mean $\mathbb{E}(X \mid Y)$.

(d) Find $\mathbb{E}(X)$ in two ways: first, using the marginal in part (b), then, using the conditional in part (c).

4. The distribution of X given a parameter Y is Exponential with the rate Y, that is, $f(x \mid Y = y) = ye^{-yx}, x > 0$. Also, Y has a prior distribution that's Uniform on $[1, 2]$. Use conditioning to find $\mathbb{E}(X)$ and $P(X > 1)$.

5. Consider a Poisson process $X(t)$ with intensity λ, so that

$$p_k(t) = P(X(t) = k) = \exp(-\lambda t)\frac{(\lambda t)^k}{k!}.$$

Let W_k be the time when kth event happens.

(a) Given that $X(4) = 5$, show that the number of events on the interval $(1, 3]$ follows a Binomial distribution, and find its parameters.

(b) Given that $X(4) = 5$, decribe the distribution of W_3 and find its expected value.

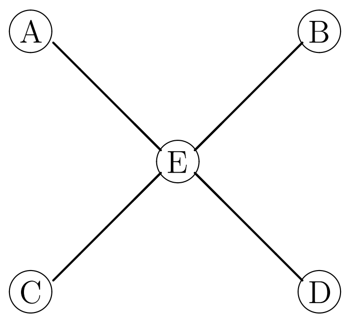
6. A Markov chain is defined by a random walk on the graph pictured below. From a given node, you are equally likely to go to any neighboring node.

(a) Specify the transition matrix.

(b) Find the stationary distribution for this Markov chain.

(c) Find the probability, when starting from A, to visit C before you visit D.

(d) Find the expected time, when starting from A, to visit C.



7. Let $X(t)$ be a birth-and-death process with values in $\{0, 1, 2\}$, the initial value $X(0) = 0$ the birth rates $\lambda_0 = \lambda_1 = 1$ and the death rates $\mu_1 = 2, \mu_2 = 3$. Let $P_n(t) = P(X(t) = n)$. Find a system of differential equations for $P_n(t), n = 0, 1, 2$ and show that their solution is

$$\begin{cases} P_0(t) = 3/5 + \exp(-2t)/3 + \exp(-5t)/15 \\ P_1(t) = 3/10 - \exp(-2t)/6 - (2/15)\exp(-5t) \\ P_2(t) = 1 - P_0(t) - P_1(t) \end{cases}$$