Ph.D. Preliminary Examination in Numerical Analysis
Department of Mathematics
New Mexico Institute of Mining and Technology
March 10, 2020, 8 AM – 12 PM

1. This exam is four hours long.
2. Work out all six problems.
3. Start the solution of each problem on a new page.
4. Number all of your pages.
5. Sign your name on the following line and put the total number of pages.
6. Use this sheet as a coversheet for your papers.

NAME: ______________________________ **No. of pages:**__________

Problem 1.

Describe the Cholesky, LU, and QR decompositions of a square matrix A. State theorems on existence and uniqueness of these decompositions (do not prove the theorems). Describe how these methods can be used to solve a linear system $Ax = b$ with an appropriate matrix A.

Problem 2.

Let A be an m by n matrix with columns A_1, A_2, ..., A_n. Let

$$\|A\|_{2,1} = \sum_{j=1}^{n} \|A_j\|_2.$$

That is, $\|A\|_{2,1}$ is the sum of the 2-norms of the columns of A. It can be shown that $\|A\|_{2,1}$ is a norm. Show that norm $\|A\|_{2,1}$ is sub-multiplicative; that is, for any A and B of compatible sizes,

$$\|AB\|_{2,1} \leq \|A\|_{2,1}\|B\|_{2,1}.$$

Hints:

a) Using the representation $AB = [AB_1, \ldots, AB_n]$, prove $\|AB\|_{2,1} \leq \|A\|_2\|B\|_{1,2}$;

b) Prove $\|A\|_2 \leq \|A\|_{2,1}$ by first proving

$$\|Ax\|_2 \leq \|A\|_{2,1}, \ \forall x \in R^n, \ \|x\|_2 \leq 1.$$

Problem 3.

Derive the Taylor series with the Lagrange remainder for $f(x) = \ln(1 + x)$ in powers of $(x - 1)$. Determine a number n such that n terms of this Taylor's series will ensure an approximation to $\ln 4$ with absolute error less than 10^{-2}.

Problem 4.

a) Describe the forward and the backward Euler methods for solving the initial value problem

$$\frac{dy}{dt} = f(t, y), \ y(0) = y_0.$$

b) Find and sketch the regions of absolute stability of the both methods. Which method, if any, is A-stable?

Hint: Apply the method to the initial-value problem

$$\frac{dy}{dt} = \lambda y(t), \quad y(0) = y_0.$$

Problem 5.

Consider the multistep method

$$U_{i+1} + \frac{3}{2}U_i - 3U_{i-1} + \frac{1}{2}U_{i-2} = 3hf(t_i, U_i)$$

for solving the initial value problem

$$u' = f(t, u),\ \ u(0) = u_0.$$

a) Determine if the method is consistent. Find the local truncation error.

b) Analyze the method for stability and convergence.

Problem 6.

Find the quadrature formula

$$Q(f) = c\,(f(x_1) + f(x_2) + f(x_3))$$

to approximate the integral $\int_{-1}^{1} f(x)\ d$ with the highest degree of precision.