PhD Preliminary Examination in Analysis

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1. Let $\alpha : [0,1] \to \mathbb{R}$ and $f : [0,1] \to \mathbb{R}$ be two real-valued functions on the interval $[0,1]$ and

$$M = \sup_{x\in[0,1]} f(x) \qquad \text{and} \qquad m = \inf_{x\in[0,1]} f(x).$$

Suppose that α is increasing and f is Riemann-integrable with respect to α on $[0,1]$. Prove that there exists a number $c \in [m, M]$ such that

$$\int_0^1 f(x) d\alpha(x) = c \int_0^1 d\alpha(x).$$

2. Let $f : [0,1] \to \mathbb{R}$ be a real-valued function on the interval $[0,1]$. Let $g : (0,1) \to \mathbb{R}$ be the function on $(0,1)$ defined for any $x \in (0,1)$ by

$$g(x) = \frac{f(x)}{x} .$$

Suppose that:

   (a) f is continuous on $[0,1]$,
   (b) f is differentiable on $(0,1)$,
   (c) f' is increasing on $(0,1)$, and
   (d) $f(0) = 0$.

Prove that g is increasing on $(0,1)$.

3. Let $\alpha \in \mathbb{R}_+$ be a positive real number. Let $x_1 \in \mathbb{R}$ be a real number such that $x_1 > \sqrt{\alpha}$. Let $(x_n)_{n=1}^{\infty}$ be a sequence defined recursively by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right), \qquad n \geq 1 .$$

Prove that the sequence $(x_n)_{n=1}^{\infty}$ is convergent.

4. Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on the interval $[0,1]$ defined for $x \in [0,1]$ by

$$f_n(x) = n^2 x (1 - x^2)^n .$$

Is the sequence $(f_n)_{n=1}^{\infty}$ uniformly convergent on $[0,1]$? Justify your answer.

5. Let f and g be two entire functions. Suppose that:

   (a) the functions f and g have no zeros, and

   (b)
$$\lim_{z\to\infty} \frac{f(z)}{g(z)} = 1\,.$$

   Show that they are the same function, that is, $f(z) = g(z)$ for any $z \in \mathbb{C}$.

6. Let C be the a sufficiently small simple closed contour not passing through the origin and n be an integer. Evaluate the integral
$$I_n = \oint\limits_C \frac{dz}{z}\left(z + \frac{1}{z}\right)^n\,.$$

   Consider the cases $n > 0, n = 0, n < 0$.

7. Let D be a simply connected domain, C be a simple closed contour in D and $a \in \mathbb{C}$ be a complex number. Let f be a function. Suppose that

   (a) f is analytic in D,

   (b) $f(z) \neq a$ for any $z \in C$.

   Show that
$$\frac{1}{2\pi i}\oint\limits_C dz \frac{f'(z)}{f(z) - a} = N,$$

   where N is the number of points z inside C such that $f(z) = a$.

8. Let f be an entire function. Suppose that $\operatorname{Im} f(z) \leq 0$ for any $z \in \mathbb{C}$. Prove that f is constant.

9. Let $\alpha, \beta \in \mathbb{R}$ be two real numbers. Use principal value integrals to show that
$$\int\limits_0^\infty \frac{dx}{x^2}\left[\cos(\alpha x) - \cos(\beta x)\right] = -\frac{\pi}{2}\left(|\alpha| - |\beta|\right)\,.$$